Identification and Calculation of the Universal Maximum Drag Reduction Asymptote by Polymers in Wall Bounded Turbulence

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Drag reduction by polymers in wall turbulence is bounded from above by a universal maximal drag reduction (MDR) velocity profile that is a log-law, estimated experimentally by Virk as $V^+(y^+) \approx 11.7 \log y^+ - 17$. Here $V^+(y)$ and y^+ are the mean streamwise velocity and the distance from the wall in "wall" units. In this Letter we propose that this MDR profile is an edge solution of the Navier-Stokes equations (with an effective viscosity profile) beyond which no turbulent solutions exist. This insight rationalizes the universality of the MDR and provides a maximum principle which allows an ab-initio calculation of the parameters in this law without any viscoelastic experimental input.

arXiv:nlin/0505010v1 [nlin.CD] 4 May 2005

The mean streamwise velocity profile in Newtonian turbulent flows in channel geometries satisfies the classic von-Kármán "log-law of the wall" which is written in wall units as

$$V^+(y^+) = \kappa_{\rm K}^{-1} \ln y^+ + B\,, \quad \text{for } y^+ \gtrsim 30\,. \tag{1}$$

Here x, y and z are the streamwise, wall-normal and spanwise directions respectively [1]. The wall units are defined as follows: let p' be the fixed pressure gradients $p' \equiv -\partial p/\partial x$, and L the mid-height of the channel. Then the Reynolds number $\mathcal{R}e$, the normalized distance from the wall y^+ and the normalized mean velocity $V^+(y^+)$ (which is in the x direction with a dependence on y only) are defined by

$$\mathcal{R}e \equiv L\sqrt{p'L}/\nu_0\ ,\ y^+ \equiv y\mathcal{R}e/L\ ,\ V^+ \equiv V/\sqrt{p'L}\ , \tag{2}$$

where ν_0 is the kinematic viscosity. The law (1) is universal, independent of the nature of the Newtonian fluid; it is one of the shortcomings of the theory of wall-bounded turbulence that the von-Kármń constant $\kappa_K \approx 0.436$ and the intercept $B \approx 6.13$ are only known from experiments and simulations [1, 2].

One of the most significant experimental findings [3] concerning turbulent drag reduction by polymers is that in channel and pipe geometries the velocity profile (with polymers added to the Newtonian fluid) is bounded between von-Kármán's log-law and another log-law which describes the maximal possible velocity profile (Maximum Drag Reduction, MDR) [4, 5, 6, 7],

$$V^+(y^+) = \kappa_{\rm V}{}^{-1} \ln\left(e\,\kappa_{\rm V} y^+\right) \quad \text{for } y^+ \gtrsim 10\,. \tag{3}$$

This law, which had been discovered experimentally by Virk (and hence the notation $\kappa_{\rm V}$), is also claimed to be universal, independent of the Newtonian fluid and the nature of the polymer additive, including flexible and rigid polymers [8]. The numerical value of the coefficient $\kappa_{\rm V}$ is presently known only from experiments, $\kappa_{\rm V}{}^{-1} \approx 11.7$, giving a phenomenological MDR law in the form [3]

$$V^+(y^+) = 11.7 \ln y^+ - 17\,. \tag{4}$$

For sufficiently high values of $\mathcal{R}e$ and concentration of the polymer, the velocity profile in a channel is expected to follow the law (3). For finite $\mathcal{R}e$, finite concentration and finite extension of the polymers one expects cross-overs back to a velocity profile parallel to the law (1), but with a larger mean velocity (i.e. with a larger value of the intercept B). The position of the cross-overs are not universal in the sense that they depend on the nature of the polymers and the flow conditions; the cross-overs are discussed in [7, 9].

While we still cannot predict from first principles the parameters in von-Kármán's log-law, the aim of this Letter is to identify the MDR log-law as an edge turbulent state in wall bounded flows, leading to a derivation of the parameters appearing in Eq. (4) without any viscoelastic input. The derivation follows from the theory of drag reduction by polymers that was developed recently, and therefore a short summary of the main aspects of the theory is in order.

Wall bounded turbulence in Newtonian fluids is discussed [1, 10] by considering the fluid velocity $\boldsymbol{U}(\boldsymbol{r})$ as a sum of its average (over time) and a fluctuating part:

$$\boldsymbol{U}(\boldsymbol{r},t) = \boldsymbol{V}(y) + \boldsymbol{u}(\boldsymbol{r},t)\ , \quad \boldsymbol{V}(y) \equiv \langle \boldsymbol{U}(\boldsymbol{r},t)\rangle\ . \tag{5}$$

The objects that enter the Newtonian theory are the mean shear $S(y)$, the Reynolds stress $W(y)$ and the kinetic energy $K(y)$:

$$S(y) \equiv dV(y)/dy\ ,\ W(y) \equiv -\langle u_x u_y\rangle\ ,\ K(y) = \langle |\boldsymbol{u}|^2\rangle/2\,. \tag{6}$$

In the presence of dilute polymers added to the Newtonian fluid one needs to complement these statistical objects with the dimensionless "conformation tensor" $\boldsymbol{R}(\boldsymbol{r},t)$ which stems from the ensemble average of the dyadic product of the end-to-end distance of the polymer chains, normalized by its equilibrium value ρ_0^2 [11, 12]. The way that the conformation tensor appears in the additional stress tensor which appears in the viscoelastic equations of motion is model dependent; it is different for flexible and rigid polymers, and it also depends on the actual model of the polymers. Nevertheless it was

shown theoretically [7, 13] that both for rigid and flexible polymers one can write down eventually the balance equations for momentum and energy at distance y away from the wall as

$$\nu(y)S(y)+W(y) \;=\; p'L\ , \quad y\ll L\ , \qquad (7)$$

$$\Big[\nu(y)\frac{a^2}{y^2}+\frac{b\sqrt{K}}{y}\Big]K(y) \;=\; W(y)S(y)\ , \qquad (8)$$

$$\nu(y) \;\equiv\; \nu_0+C\nu_p\langle R_{yy}\rangle\ . \qquad (9)$$

In Eq.(7) the right hand side is the rate at which momentum is generated by the pressure head, $W(y)$ is the momentum flux in physical space towards the wall, and $\nu(y)S(y)$ stands for the Newtonian viscous dissipation of momentum in addition to the polymer contribution to the dissipation of momentum. The effective viscosity $\nu(y)$ is given by Eq. (9) where ν_p is the viscosity due to the polymers in the limit of zero shear and C is a constant of the order of unity. Similarly, in Eq. (8) the first term on the left hand side is the combined Newtonian and polymer contributions to the energy dissipation, the second models the inertial energy fluxes, and the right hand side is the (exact) energy production rate. The coefficients a and b are dimensionless and of the order of unity.

As in the Newtonian case, the balance equations need to be supplemented by a relation between $K(y)$ and $W(y)$. Rigorously the Cauchy-Schwartz ineqality leads to $W(y)\le K(y)$; experimentally one find that

$$W(y)=c_{\rm V}^2K(y)\ , \qquad (10)$$

with $c_{\rm V}$ apparently y-independent outside the viscous boundary layer. To derive the functional form of the MDR [4] one asserts that the terms containing $\nu(y)$ in the balance equations (7) and (8) overwhelm the inertial terms, and then together with (10) one derives $S(y)\sim$ Const.$/y$ which is the log-law for the MDR. Consisten with this law the effective viscosiy turns to be linear in y. The increase in the viscosity of course increases the dissipation, but it was argued that the momentum flux W is reduced even further, leading to an increase in the mean momentum of the flow (for a given pressure head) , i.e. drag reduction. It is easy to argue [4] that the slope of the new log-law is larger that the slope in von-Kármán's log-law, and hence drag reduction is obtained. Nevertheless, since neither $c_{\rm V}$ nor the constants a and b in (8) are known apriori, the actual slope of the MDR could not be determined. This shortcoming is remedied in the rest of this Letter.

The crucial new insight that will explain the universality of the MDR and furnish the basis for its calculation is that the MDR is a marginal flow state of wall-bounded turbulence. In other words this is the solution of Eqs. (7) and (8) for which $S(y)$ (or equivalently, the velocity profile) is the maximal possible for any turbulent solution. Attempting to increase $S(y)$) further results in the collapse of the turbulent solutions in favor of a stable laminar solution $W=0$. As such, the MDR is universal by definition, and the only question is whether a polymeric (or other additive) can supply the particular effective viscosity $\nu(y)$ that drives Eqs. (7) and (8) to attain the marginal solution that maximizes the velocity profile. We predict that the same marginal state will exist in numerical solutions of the Navier-Stokes equations furnished with a y-dependent viscosity $\nu(y)$. There will be no turbulent solutions with velocity profiles higher than the MDR.

To see this explicitly, we first rewrite the balance equations in wall units. Define ${\delta^+}^2\equiv a^2K(y)/W(y)$, taken for simplicity as y-independent; we know from the Newtonian limit (in which $\nu(y)=\nu_0$) that $\delta^+\approx 6$ [14]. Once we write the equations with $\nu(y)$ the ratio a^2K/W can change drastically, and we denote it below by Δ^2. With $S^+\equiv S\nu_0/p'L$, $K^+\equiv K/p'L$, $W^+\equiv W/p'L$ and $\nu^+(y^+)\equiv\nu(y^+)/\nu_0$, The balance equations are written as

$$\nu^+(y^+)S^+(y^+)+W^+(y^+)=1\ , \qquad (11)$$

$$\nu^+(y^+)\frac{\Delta^2}{{y^+}^2}+\frac{\sqrt{W^+}}{\kappa_{\rm K}y^+}=S^+\ . \qquad (12)$$

In Eq. (12) $\Delta\to\delta^+$ when $\nu^+(y^+)\to 1$ (the Newtonian limit). The bunch of numerical constants in the second term on the LHS of (12) was replaced with $\kappa_{\rm K}^{-1}$ in agreement with newtonian log-law when $\nu^+(y^+)\to 1$. In fact the second term on the left hand side of Eq.(12) (which vanishes at the MDR) contains a factor $(\Delta/\delta^+)^3$. This factor is omitted for simplicity; accounting for this factor complicates slightly the algbra, leaving the final conclusions unchanged. Substituting now S^+ from Eq. (11) into Eq. (12) leads to a quadratic equation for $\sqrt{W^+}$. This equation has as a zero solution for W^+ (laminar solution) as long as $\nu^+(y^+)\Delta/y^+=1$. Turbulent solutions are possible only when $\nu^+(y^+)\Delta/y^+<1$. Thus at the edge of existence of turbulent solutions we find $\nu^+\propto y^+$. This is not surprising, since it was observed already in previous work that the MDR solution is consistent with an effective viscosity which is asymptotically linear in y^+ [4, 5]. It is therefore sufficient to seek the edge solution by maximizing the velocity profile with respect to linear viscosity profiles, and we rewrite Eqs. (11) and (12) with an effective viscosity that depends linearly on y^+ outside the boundary layer of thickness δ^+:

$$[1+\alpha(y^+-\delta^+)]S^++W^+=1\ , \qquad (13)$$

$$[1+\alpha(y^+-\delta^+)]\frac{\Delta^2(\alpha)}{{y^+}^2}+\frac{\sqrt{W^+}}{\kappa_{\rm K}y^+}=S^+\ . \qquad (14)$$

We now endow Δ with an explicit dependence on the slope of the effective viscosity, $\Delta(\alpha)\to\delta^+$ when $\alpha\to 0$. For $\alpha\neq 0$ the ratio a^2K/W is expected to depend on α (drag reduction involves a reduction in W), and this dependence is an important part of the following theory. We can present $\Delta(\alpha)$ in terms of a dimensionless scaling function $f(x)$,

$$\Delta(\alpha)=\delta^+f(\alpha\delta^+)\ . \qquad (15)$$

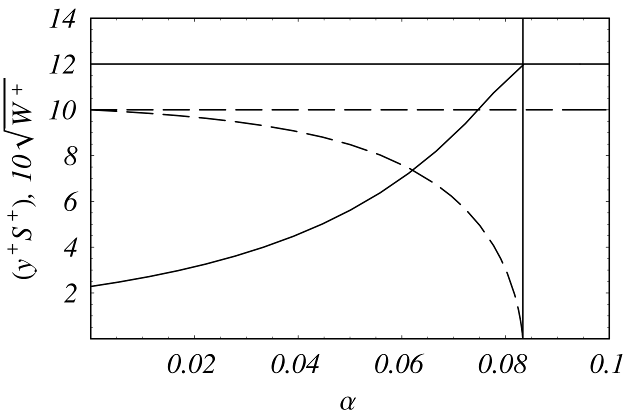

FIG. 1: The solution for $10\sqrt{W^+}$ (dashed line) and y^+S^+ (solid line) in the asymptotic region $y^+ \gg \delta^+$, as a function of α. The vertical solid line $\alpha = 1/2\delta^+ = 1/12$ which is the edge of turbulent solutions; Since $\sqrt{W^+}$ changes sign here, to the right of this line there are only laminar states. The horizontal solid line indicates the highest attainable value of the slope of the MDR logarithmic law $1/\kappa_{\rm V} = 12$.

Obviously, $f(0) = 1$. In the Appendix we show that the balance equation (13) and (14) (with the prescribed form of the effective viscosity profile) have an non-trivial symmetry that leaves them invariant under rescaling of the wall units. This symmetry dictates the function $\Delta(\alpha)$ in the form

$$\Delta(\alpha) = \frac{\delta^+}{1-\alpha\delta^+} \ . \tag{16}$$

Armed with this knowledge we can now find the maximal possible velocity far away from the wall, $y^+ \gg \delta^+$. There the balance equations simplify to

$$\alpha y^+ S^+ + W^+ = 1 \ , \tag{17}$$

$$\alpha\Delta^2(\alpha) + \sqrt{W^+}/\kappa_{\rm K} = y^+ S^+ \ . \tag{18}$$

These equations have the y^+-independent solution for $\sqrt{W^+}$ and y^+S^+:

$$\begin{aligned} \sqrt{W^+} &= -\frac{\alpha}{2\kappa_{\rm K}} + \sqrt{\left(\frac{\alpha}{2\kappa_{\rm K}}\right)^2 + 1 - \alpha^2\Delta^2(\alpha)} \ , \\ y^+S^+ &= \alpha\Delta^2(\alpha) + \sqrt{W^+}/\kappa_{\rm K} \ . \end{aligned} \tag{19}$$

Obviously, (see Fig. 1), the supremum of y^+S^+ is obtained when W^+ vanishes, which happens precisely when $\alpha = 1/\Delta(\alpha)$. Using Eq. (16) we find the solution $\alpha = \alpha_m = 1/2\delta^+$. Then $y^+S^+ = \Delta(\alpha_m)$, giving $\kappa_{\rm V}^{-1} = 2\delta^+$. Using the estimate $\delta^+ \approx 6$ we get the final prediction for the MDR. Using Eq. (3) with $\kappa_{\rm V}^{-1} = 12$, we get

$$V^+(y^+) \approx 12 \ln y^+ - 17.8 \ . \tag{20}$$

This result is in close agreeement with the empirical law (4) proposed by Virk. Note that the numbers appearing in Virk's law correspond to $\delta^+ = 5.85$, which is well within the error bar on the value of this Newtonian parameter. Note that we can easily predict where the asymptotic law turns into the viscous layer upon the approach to the wall. We can consider an infinitesimal W^+ and solve Eqs. (11) and (12) for S^+ and the viscosity profile. The result, as before, is $\nu^+(y) = \Delta(\alpha_m)y^+$. Since the effective viscosity cannot fall bellow the Newtonian limit $\nu^+ = 1$ we see that the MDR cannot go below $y^+ = \Delta(\alpha_m) = 2\delta^+$. We thus expect an extension of the viscous layer by a factor of 2.

One should note that the result $W^+ = 0$ should not be interpreted as $W = 0$. The differeence between the two objects is the factor of $\mathcal{R}e^2$, $W \propto \mathcal{R}e^2 W^+$. Since the MDR is reached asymptotically as $\mathcal{R}e \to \infty$, there is enough turbulence at this state to stretch the polymers to supply the needed effective viscosity. Nevertheless our discussion is in close correspondence with the experimental remark by Virk [3] that close to the MDR asymptote the flow appears laminar.

In summary, one can probably improve further the model of the Newtonian wall-bounded flow, making it more elaborate and more precise. But the message of this Letter will stay unchanged; whatever is the model of choice, once endowed with effective viscosity $\nu(y)$ instead of ν_0, there would exist a profile of $\nu(y)$ that would result in a maximal possible velocity profile at the edge of existence of turbulent solutions. That profile is the prediction of the said model of choice for the MDR. In particular we offer a prediction for simulations: direct numerical simulations of the Navier-Stokes equations in a channel, endowed with a linear viscosity profile [5], will not be able to support turbulent solutions when the slope of the viscosity profile exceeds the critical value that is in correspondence with the slope of the MDR. Notwithstanding, it is gratifying to discover that even a simple model of the balance of energy and momentum is sufficient, in light of the insight presented in this Letter, to predict ab-initio the functional form *and* the parameters that determine the Maximum Drag Reduction asymptote.

APPENDIX A: THE SCALING FUNCTION

Consider the following identity:

$$\begin{aligned} \nu^+(y^+) &= 1 + \alpha(y^+ - \delta^+) \\ &= [1 + \alpha(y^+ - \tilde\delta) + \alpha(\tilde\delta - \delta^+)] \\ &= g(\tilde\delta)\left[1 + \frac{\alpha}{g(\tilde\delta)}(y^+ - \tilde\delta)\right] \ , \end{aligned} \tag{A1}$$

where

$$g(\tilde\delta) \equiv 1 + \alpha(\tilde\delta - \delta^+) \ , \quad \tilde\delta \ge \delta^+ \ . \tag{A2}$$

Next introduce newly renormalized units using the effective viscosity $g(\tilde\delta)$, i.e.

$$y^\ddagger \equiv \frac{y^+}{g(\tilde\delta)}, \quad \delta^\ddagger \equiv \frac{\tilde\delta}{g(\tilde\delta)}\ , \quad S^\ddagger \equiv S^+ g(\tilde\delta)\ , \quad W^\ddagger \equiv W^+\ . \tag{A3}$$

In terms of these variables the balance equations are rewritten as

$$[1+\alpha(y^\ddagger-\delta^\ddagger)]S^\ddagger + W^\ddagger = 1\ , \tag{A4}$$

$$[1+\alpha(y^\ddagger-\delta^\ddagger)]\frac{\Delta^2(\alpha)}{{y^\ddagger}^2} + \frac{\sqrt{W^\ddagger}}{\kappa_{\rm K} y^\ddagger} = S^\ddagger\ . \tag{A5}$$

These equations are isomorphic to (13) and (14) with δ^+ replaced by $\delta^\ddagger$. The ansatz (15) is then replaced by

$$\Delta(\alpha) = \frac{\delta^+}{g(\tilde\delta)} f(\alpha\delta^\ddagger)\ . \tag{A6}$$

This form is dictated by the following considerations: (i) $\Delta(\alpha) \to \delta^+$ when $\alpha \to 0$, (ii) all lengths scales in the rescaled units are divided by $g(\tilde\delta)$, and thus the pre-factor in front of f becomes $\delta^+/g(\tilde\delta)$, and (iii) $\alpha\delta^+$ in Eq. (12) is now replaced in Eq. (A5) by $\alpha\delta^\ddagger$, leading to the new argument of f.

Since the function $\Delta(\alpha)$ cannot change due to the change of variables, the function $\Delta(\alpha)$ given by Eq. (A6) should be identical to that given by Eq. (15):

$$\delta^+ f(\alpha\delta^+) = \frac{\delta^+}{g(\tilde\delta)} f(\alpha\delta^\ddagger)\ . \tag{A7}$$

Using the explicit form of $g(\tilde\delta)$ Eq. (A2), and choosing (formally first) $\tilde\delta = \tilde\delta^\ddagger = 0$ we find that $f(\xi) = 1/(1-\xi)$. It is easy to verify that this is indeed the solution of the above equation for any value of $\delta^\ddagger$, and therefore the unique form of Eq. (16).

ACKNOWLEDGMENTS

We thank T.S. Lo and Anna Pomyalov for very useful discussions, and for careful reading of early versions of the manuscript. This work has been supported in part by the US-Israel Binational Science Foundation, by the European Comission under a TMR grant, and by the Minerva Foundation, Munich, Germany.

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